Group Geodesic Growth

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Notation

Volume Growth

Given a group, G, with some finite symmetric generating set, $X = X^{-1}$, we can define the length of any element $g \in G$ as the length of the shortest word $w \in X^\star$ equivalent to g.

$$\|g\|_X \coloneqq \min\{n \in \mathbb{N} \mid x_1 x_2 \cdots x_n =_G g \text{ with each } x_j \in X\}$$

Then, G becomes a homogeneous geodesic metric space with metric

$$d_X(g, h) \coloneqq \left\|gh^{-1}\right\|_X$$

Then, the (volume) growth function $f_X(n)$ will be the volume of a closed ball of radius n. That is,

$$f_X(n) \coloneqq \#\left|\{g \in G \,:\, \|g\|_X \leqslant n\}\right|$$

Similarly, spherical growth will be defined as

$$s_X(n) \coloneqq \#\left|\{g \in G \,:\, \|g\|_X = n\}\right|$$

Usual Growth Relation

Then, defining a partial order on such growth functions as

$$f \preccurlyeq g \iff \text{there exists a } C \in \mathbb{N}_+ \text{ s.t. } f(n) \leqslant C \cdot g(C \cdot n)$$

It is a classic result by Švarc[1] and Milnor[2] that the corresponding equivalence relation

$$f \sim g \iff f \preccurlyeq g \text{ and } g \preccurlyeq f$$

is invariant under quasi-isometry. In particular, for any two symmetric generating sets X and Y of G it is true that $f_X \sim f_Y$.

[1] Švarc, 'A volume invariant of coverings', 1955.

[2] Milnor, 'A note on curvature and fundamental group', 1968.

Growth Types

Then, the growth function, f, of a group is called:

Polynomial: If there exists an $d \in \mathbb{R}$ such that $f \preccurlyeq n^d$ *(the smallest such d is called the degree)*

Exponential: If it is true that $e^n \preccurlyeq f$

Intermediate: If it is neither polynomial nor exponential

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In 1968, Milnor asked the following question[3]

Are all group growth functions either polynomial of integer degree or exponential?

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The first is a corollary to Gromov theorem[5] and that the all virtually nilpotent groups have growth of integer degree[6].

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The second half was resolved by Grigorchuk when he showed a particular example to have intermediate growth[7].

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Geodesic Growth

Definition

Given a finite symmetric generating set, X, the geodesic growth function, $F_X(n)$, will count the number of geodesics words within a ball of size n starting from its centre. That is,

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What's different?

No longer invariant under change of generating set!!!

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Question: Does there exist a group with a generating set such that the geodesic growth function is intermediate? What class of usual growth do they possess?

Main Question of Focus

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Natural starting point: Grigorchuk's group with usual generators

[8] Brönnimann, 'Geodesic growth of groups', 2016.

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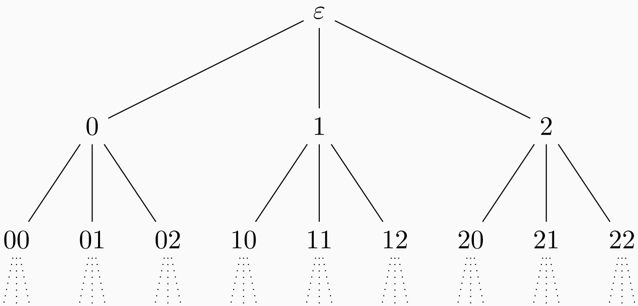
Fabrykowski-Gupta Group

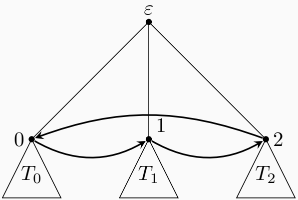
Regular Rooted Tree

Each of its elements is an automorphism on a 3-regular rooted tree

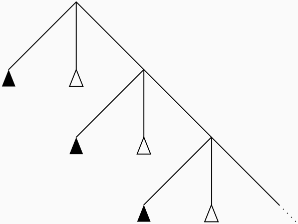


Generators of Fabrykowski-Gupta

In particular, the group is generated by actions a and b where



Action of a



Action of b

where ▲ is an action by a and $\triangle$ is the trivial action

Previous

Experimental Computation

In Honours[10] Thesis: Attempted to explicitly compute the geodesics of the Fabrykoswki-Gupta group in the hopes of finding an exponential growth sub-family or some ‘obvious’ pattern help realise intermediate geodesic growth

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$$\text{brute-force} \sim O\left(\sum_{\ell=1}^{n} s(\ell-1)^2 \; \ell \log \ell\right)$$

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$$O\left(\sum_{\ell=1}^{n} s(\ell-1)\left(\log s(\ell-1)\right)^2 \;\ell\log\ell\right)$$

(2) Implementing in C (later in C++11) with multithreading

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Remarks:

(1) In the case of Fabrykowski-Gupta, the complexity is only a polynomial times the theretic optimal
(2) Generalisable to the class of contracting groups[11]

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'Results': Inconclusive ☹

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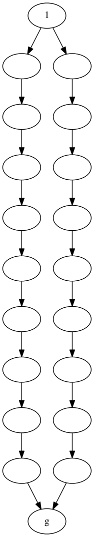
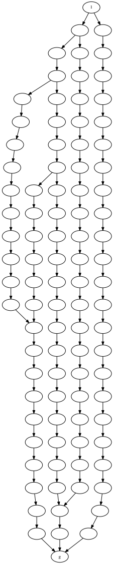
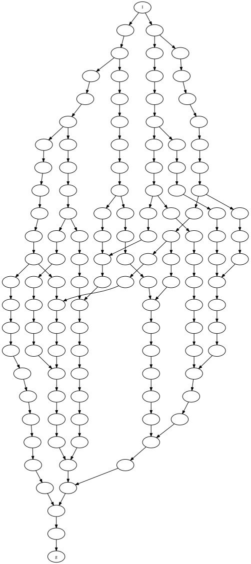
Although, some pretty pictures







Future

What to try

- More meaningful experimental computational
- Attempt to find some formal language which includes the geodesics of Fabrykowski-Gupta
- Consider different groups of intermediate growth: Brönnimann's thesis[12] was not exhaustive
- Consider virtually nilpotent groups that are not known a priori to have only polynomial or exponential growth

[12] Brönnimann, 'Geodesic growth of groups', 2016.

References

Bass, Hyman. 'The degree of polynomial growth of finitely generated nilpotent groups'. In: *Proc. London Math. Soc. (3)* 25 (1972), pp. 603–614. ISSN: 0024-6115. URL: https://doi.org/10.1112/plms/s3-25.4.603.

Brönnimann, Julie Marie. 'Geodesic growth of groups'. PhD thesis. Université de Neuchâtel, 2016. URL: http://doc.rero.ch/record/277391/files/00002547.pdf.

Carlitz, L. et al. 'Advanced Problems: 5600-5609'. In: *The American Mathematical Monthly* 75.6 (1968), pp. 685–687. ISSN: 00029890, 19300972. URL: http://www.jstor.org/stable/2313822.

Fabrykowski, Jacek and Narain Gupta. 'On groups with sub-exponential growth functions'. In: *J. Indian Math. Soc. (N.S.)* 49.3-4 (1985), 249–256 (1987). ISSN: 0019-5839.

Grigorchuk, Rostislav. 'Degrees of growth of finitely generated groups and the theory of invariant means'. In: *Izv. Akad. Nauk SSSR Ser. Mat.* 48.5 (1984), pp. 939–985. ISSN: 0373-2436.

Gromov, Mikhael. 'Groups of polynomial growth and expanding maps'. In: *Inst. Hautes Études Sci. Publ. Math.* 53 (1981), pp. 53–73. ISSN: 0073-8301. URL: http://www.numdam.org/item?id=PMIHES_1981__53__53_0.

Milnor, John. 'A note on curvature and fundamental group'. In: *J. Differential Geometry* 2 (1968), pp. 1–7. ISSN: 0022-040X. URL: http://projecteuclid.org/euclid.jdg/1214501132.

Švarc, A. S. 'A volume invariant of coverings'. Russian. In: *Dokl. Akad. Nauk SSSR (N.S.)* 105 (1955), pp. 32–34. ISSN: 0002-3264.

Thanks!