Group Geodesic Growth

Alex Bishop

University of Technology Sydney, Australia

Introduction

Topics of regular group growth have been well-studied; for example, Gromov's famous theorem that the groups of polynomial growth are precisely the virtually nilpotent ones, a corollary to which being that only polynomial growth of integer degree is possible; and Grigorchuk's proof of a group with intermediate regular growth. However, if these problems are slightly modified to instead count the number of shortest paths (geodesics) in a group, then all of these well-known results become open questions.

Def$^{\mathrm{n}}$: Geodesic Growth

Given a group G with finite symmetric generating set X, the *geodesic growth function*, $\Gamma_{G,X}$, counts the geodesic word within a given length

$$\Gamma_{G,X}(n) \coloneqq |\{w = x_1 x_2 \cdots x_k \in X^* \mid \ell_X(w) = k \leqslant n\}|$$

Then, given the regular growth function, $\gamma_{G,X}$, it is clear that

$$\gamma_{G,X}(n) \leqslant \Gamma_{G,X}(n) \leqslant |X| \left(|X|-1\right)^{n-1}$$

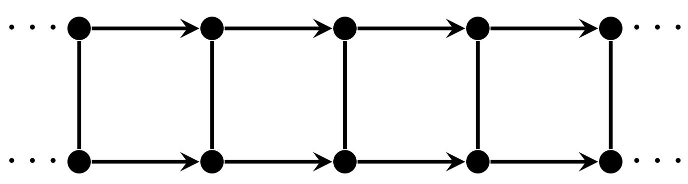
Example 2: Different Geodesic Growth Classes

Presentation:

$$\langle a, t \mid t^2, [a,t] \rangle$$

Geodesic Growth Rate: *polynomial*

$$\Gamma_{\{a^{\pm 1}, t\}}(n) = n^2 + 3n \quad \textit{(for } n \geqslant 2\textit{)}$$

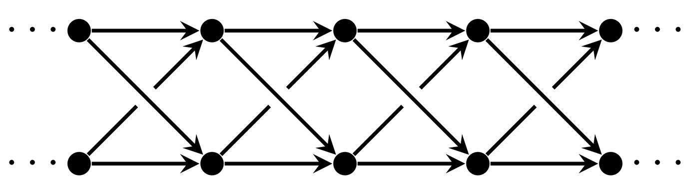


Tietze Transform: (where $c = at$)

$$\langle a, c \mid a^2 = c^2, [a,c] \rangle$$

Geodesic Growth Rate: *exponential*

$$\Gamma_{\{a^{\pm 1}, c^{\pm 1}\}}(n) = 2^{n+1} - 1$$

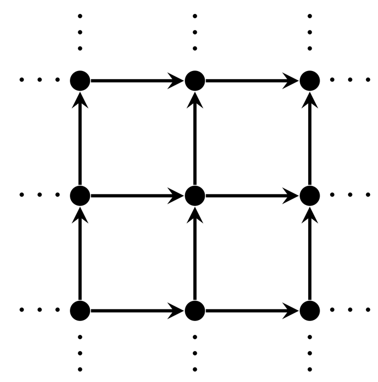


Example 1: $\mathbb{Z}^2$

Presentation: $\mathbb{Z}^2 = \langle a, b \mid [a,b] \rangle$

Regular Growth: *polynomial* — $\gamma(n) = 2n^2 + 2n + 1$

Geodesic Growth: *exponential* — $\Gamma(n) = 2^{n+3} - 4n - 7$



Proposition: (Bridson, Burillo, Elder and Šunić 2012)

$\mathbb{Z}^2$ has exponential geodesic growth with respect to every finite generating set

Example 3: Virtually $\mathbb{Z}^2$

Presentation: $\langle a, b, t \mid [a,b], t^2, a^t = b \rangle$

Geodesic Growth: *exponential* (it contains a $\mathbb{Z}^2$, see Example 1)

$$\Gamma_{\{a^{\pm 1}, b^{\pm 1}, t\}}(n) = (n+1) \cdot 2^{n+2} - 2n^2 - 6n - 2 \qquad \textit{(for } n \geqslant 5\textit{)}$$

Remove b by a Tietze Transform: $\langle a, t \mid [a, a^t], t^2 \rangle$

Geodesic Growth: *polynomial* (compare this with Example 1)

$$\Gamma_{\{a^{\pm 1}, t\}}(n) = \left(2n^3 - 2n + 18\right)/3 \qquad \textit{(for } n \geqslant 5\textit{)}$$

Main Question

Does there exist a group with intermediate geodesic growth?

How about Grigorchuk's Group?

Consider the n^{th} level Schreier Graphs

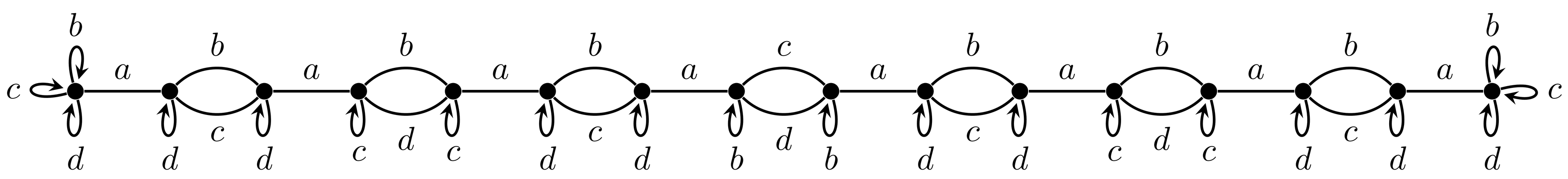
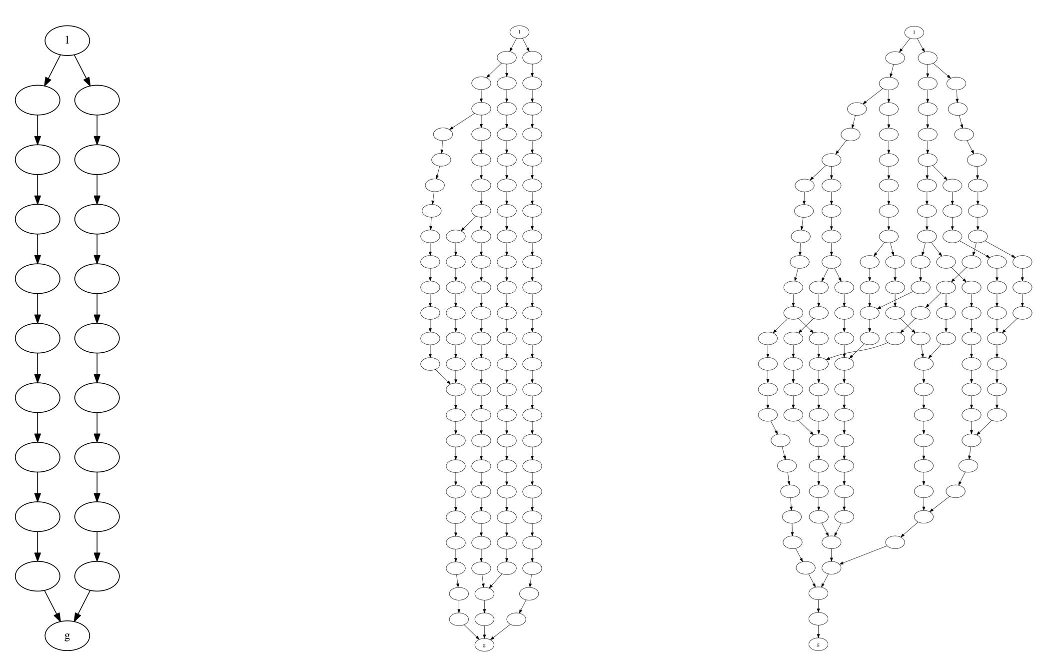


Fig. 4: 4^{th} level Schreier graph

Anwser: exponential — There are exponentially many geodesics in the Schreier graphs, which are also geodesics of the group.

Experimental Mathematics

Results of a computer enumeration of geodesics (of the Gupta-Fabrykowski group):



What about groups similar to Grigorchuk's?

Following the idea of considering the Schreier graphs Brönnimann showed

Grigorchuk G_ω: exponentially many geodesics

Gupta-Sidki p-groups: exponentially many geodesics

Square group: exponentially many geodesics

Spinal group: exponentially many geodesics

Gupta-Fabrykowski: technique is inconclusive; unkown if exponential

Proposition: (Bartholdi and Grigorchuk 2000)

There are only polynomially many (of degree $\log_2 3$) geodesics in the level Schreier graphs of the Gupta-Fabrykowski group

Potential Techniques

Given a particular group of intermediate regular growth:

- Consider the function which counts the number of geodesics per element. If this function has a sub-exponential upper bound with respect to the word length then we have intermediate geodesic growth
- Consider different generating sets e.g. does Grigorchuk's group have exponential geodesic growth with respect to every generating set?
- Attempt to find a convenient formal language which includes all the geodesics.

or attempt to cosntruct a virtually nilpotent group with the desired property